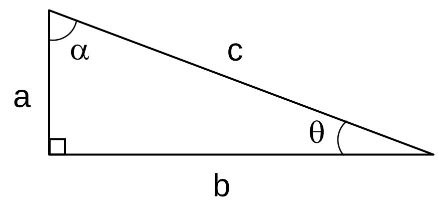
TRIGONOMETRIC FUNTIONS AND IDENTITIES



It must be noted that the three functions are the reciprocals of the other three,

Fundamental Relations

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Other relations are readily proved and these are the following;

Reciprocal relations

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Square relations

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Prove the following identities:

1. $\cos\theta\tan\theta = \sin\theta$
   - to reduce the left to that of the right, use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

   $$\frac{\cos\theta\sin\theta}{\cos\theta} = \text{RS}$$

   then we have,

   $$\boxed{\sin\theta = \sin\theta}$$

2. $\cot\theta\cos\theta = \csc\theta - \sin\theta$
   - to reduce the right to that of the left, $\csc\theta = \dfrac{1}{\sin\theta}$

   $$\cot\theta\cos\theta = \frac{1}{\sin\theta} - \sin\theta = \frac{1-\sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta} = \frac{\cos\theta\cos\theta}{\sin\theta}$$

   but $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, substituting we have,

   $$\boxed{\cot\theta\cos\theta = \cot\theta\cos\theta}$$

3. $\cot\theta + \tan\theta = \dfrac{\csc^2\theta + \sec^2\theta}{\csc\theta\sec\theta}$

   $$= \frac{\csc^2\theta}{\csc\theta\sec\theta} + \frac{\sec^2\theta}{\csc\theta\sec\theta} = \frac{\csc\theta}{\sec\theta} + \frac{\sec\theta}{\csc\theta}$$

   but $\csc\theta = \dfrac{1}{\sin\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

   $$= \frac{\frac{1}{\sin\theta}}{\frac{1}{\cos\theta}} + \frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta}} = \left[\frac{1}{\sin\theta} * \frac{\cos\theta}{1}\right] + \left[\frac{1}{\cos\theta} * \frac{\sin\theta}{1}\right] = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

   but $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

   $$\boxed{= \cot\theta + \tan\theta}$$

4. $\dfrac{\sin\theta+\tan\theta}{\cot\theta+\csc\theta}=\sin\theta\tan\theta$

$$\frac{\sin\theta+\dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos\theta}{\sin\theta}+\dfrac{1}{\sin\theta}}=\frac{\dfrac{\sin\theta\cos\theta+\sin\theta}{\cos\theta}}{\dfrac{\cos\theta+1}{\sin\theta}}=\frac{\dfrac{(\cos\theta+1)\sin\theta}{\cos\theta}}{\dfrac{\cos\theta+1}{\sin\theta}}=\frac{(\cos\theta+1)\sin\theta}{\cos\theta}*\frac{\sin\theta}{\cos\theta+1}=\text{ RS}$$

but $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$

$$\boxed{\frac{\sin\theta\sin\theta}{\cos\theta}=\sin\theta\tan\theta}$$

5. $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2=2$

$$\left(\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\right)+\left(\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta\right)=\text{ RS}$$

$$\left(\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta\right)+\left(\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta\right)=\text{ RS}$$

$$(1+2\sin\theta\cos\theta)+(1-2\sin\theta\cos\theta)=\text{ RS}$$

$$1+2\sin\theta\cos\theta+1-2\sin\theta\cos\theta=2+2\sin\theta\cos\theta-2\sin\theta\cos\theta=\text{ RS}$$

$$\boxed{2=2}$$

6. $\sin^4\theta-\cos^4\theta=\sin^2\theta-\cos^2\theta$

$$(\sin^2\theta)^2-(\cos^2\theta)^2=\left(\sin^2\theta+\cos^2\theta\right)\left(\sin^2\theta-\cos^2\theta\right)=\text{RS}$$

$$1(\sin^2\theta-\cos^2\theta)=\boxed{\sin^2\theta-\cos^2\theta}$$

7. $\dfrac{\cos\theta}{\sec\theta}-\dfrac{\sin\theta}{\cot\theta}=\dfrac{\cos\theta\cot\theta-\tan\theta}{\csc\theta}$

$$\text{LS }=\frac{\cos\theta\dfrac{\cos\theta}{\sin\theta}-\dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\sin\theta}}=\frac{\dfrac{\cos^2\theta}{\sin\theta}-\dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\sin\theta}}=\frac{\dfrac{\cos^3\theta-\sin^2\theta}{\sin\theta\cos\theta}}{\dfrac{1}{\sin\theta}}=\frac{\cos^3\theta-\sin^2\theta}{\sin\theta\cos\theta}*\frac{\sin\theta}{1}$$

$\text{LS}=\cos^2\theta-\dfrac{\sin^2\theta}{\cos\theta}$ but $\cos\theta=\dfrac{1}{\sec\theta}$; $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ and $\tan\theta=\dfrac{1}{\cot\theta}$

$$\text{LS}=\cos\theta\cos\theta-\frac{\sin\theta\sin\theta}{\cos\theta}=\frac{\cos\theta}{\sec\theta}-\frac{\sin\theta}{\cot\theta}$$

8. $\dfrac{1}{1-\sin\theta} = \dfrac{\cot\theta}{\cot\theta - \cos\theta}$

$$\text{LS} = \frac{\dfrac{\cos\theta}{\sin\theta}}{\dfrac{\cos\theta}{\sin\theta} - \cos\theta} = \frac{\dfrac{\cos\theta}{\sin\theta}}{\dfrac{\cos\theta - \cos\theta\sin\theta}{\sin\theta}} = \frac{\dfrac{\cos\theta}{\sin\theta}}{\dfrac{(1-\sin\theta)\cos\theta}{\sin\theta}} = \frac{\cos\theta}{\sin\theta} * \frac{\sin\theta}{(1-\sin\theta)\cos\theta}$$

$$\text{LS} = \boxed{\frac{1}{1-\sin\theta}}$$

SUGGESTIONS IN PROVING IDENTITIES

1. Start work on the more complicated member of the identity.
2. If one member contains one or more indicated operations, perform the indicated operations first.
3. If one member contains more than one function while the other contains only one function, reduce the functions in the first member to the function in the second. Use the fundamental relations for this step.
4. If the numerator of one member contains several terms and the denominator contains only one functions by performing the indicated operations. Then apply the fundamental relations.
5. If either member is factorable, find the factors. In many cases, the next step is suggested by the result of the factoring operations.
6. Sometimes, it is necessary to multiply the numerator and the denominator by the same factor to obtain the desired reduction.

If none of the above steps seems applicable, express the functions in the more complicated member in terms of sine and cosine. Simplify the result afterwards.

FUNCTIONS of the SUM and DIFFERENCE of two ANGLES

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$

$$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$$

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$$

$$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$$

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

$$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$$

EXERCISES:

Page 126

1. Find $\cos 75^o$, $\tan 75^o$ and $\cot 75^o$

$\cos 75^o = \cos(30^o + 45^o)$

$= \cos 30^o \cos 45^o - \sin 30^o \sin 45^o$

$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)-\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)$ ← values from special angles

$= \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$

$= \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$

$$\tan 75^\circ = \tan(30^\circ + 45^\circ)$$

$$= \frac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ}$$

$$= \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}(1)} = \frac{\frac{\sqrt{3}+3}{3}}{\frac{3-\sqrt{3}}{3}}$$

$$\tan 75^\circ = \frac{\sqrt{3}+3}{3-\sqrt{3}}$$

$$\cot 75^\circ = \frac{1}{\tan 75^\circ}$$

$$= \frac{1}{\frac{\sqrt{3}+3}{3-\sqrt{3}}}$$

$$\cot 75^\circ = \frac{3-\sqrt{3}}{\sqrt{3}+3}$$

Simplify the expression

1. $\sin(\theta + 30) + \cos(\theta + 60)$

$$= \sin\theta\cos 30 + \cos\theta\sin 30 + \cos\theta\cos 60 - \sin\theta\sin 60$$

$$= \sin\theta\left(\frac{\sqrt{3}}{2}\right) + \cos\theta\sin\left(\frac{1}{2}\right) + \cos\theta\left(\frac{1}{2}\right) - \sin\theta\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta$$

$$= \cos\theta$$

Prove the following identities:

1. $\sin(\theta+\phi)\sin(\theta-\phi)=\sin^2\theta-\sin^2\phi$

$$[\sin\theta\cos\phi+\cos\theta\sin\phi][\sin\theta\cos\phi-\cos\theta\sin\phi] = RS$$

$$(\sin\theta\cos\phi)^2-(\cos\theta\sin\phi)^2 = RS$$

$$\sin^2\theta\cos^2\phi-\cos^2\theta\sin^2\phi = RS$$

$$\sin^2\theta(1-\sin^2\phi)-(1-\sin^2\theta)(\sin^2\phi) = RS$$

$$\sin^2\theta-\sin^2\theta\sin^2\phi-\sin^2\phi+\sin^2\theta\sin^2\phi = RS$$

$$\sin^2\theta-\sin^2\phi = RS$$

2. $\tan(45+\theta)=\dfrac{1+\tan\theta}{1-\tan\theta}$

$$\frac{\tan 45+\tan\theta}{1-\tan 45\tan\theta} = RS$$

but tan 45 = 1 ← from special angles

$$\frac{1+\tan\theta}{1-(1)\tan\theta} = RS$$

$$\frac{1+\tan\theta}{1-\tan\theta} = RS$$

FUNCTIONS of TWICE an ANGLE

sin $(\theta + \theta) = \sin\theta \cos\theta + \sin\theta \cos\theta$

If $\theta = \phi$

$\sin 2\theta = \sin\theta \cos\theta + \sin\theta \cos\theta$

$$\boxed{\mathbf{\sin 2\theta = 2 \sin\theta \cos\theta}}$$

cos $(\theta + \theta) = \cos\theta \cos\theta - \sin\theta \sin\theta$

If $\theta = \phi$

$\cos 2\theta = \cos\theta \cos\theta - \sin\theta \sin\theta$

$$\boxed{\mathbf{\cos 2\theta = \cos^2 \theta - \sin^2 \theta}}$$

$$\tan\theta + \theta = \frac{\tan\theta + \tan\theta}{1 - \tan\theta \tan\theta}$$

$$\tan 2\theta = \frac{\tan\theta + \tan\theta}{1 - \tan\theta \tan\theta}$$

$$\boxed{\mathbf{\tan 2\theta = \frac{2 \tan\theta}{1 - \tan^2 \theta}}} \qquad \boxed{\mathbf{\cot 2\theta = \frac{\cot^2 \theta - 1}{2 \cot\theta}}}$$

FUNCTIONS OF HALF AN ANGLE

$$\cos^2\phi + \sin^2\phi = 1 \qquad \text{eq. (1)}$$

$$\cos^2\phi - \sin^2\phi = \cos 2\phi \qquad \text{eq. (2)}$$

Add equations (1) and (2)

$$2\cos^2\phi = 1 + \cos 2\phi$$

$$\cos^2\phi = \frac{1+\cos 2\phi}{2};$$

$$\cos\phi = \pm\sqrt{\frac{1+\cos 2\phi}{2}}; \qquad \text{but} \quad \phi = \frac{1}{2}\theta \quad ; \qquad \boxed{\cos\frac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}}}$$

$$\sin\phi = \pm\sqrt{\frac{1-\cos 2\phi}{2}}; \qquad \boxed{\sin\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}}}$$

$$\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} = \sqrt{\frac{\frac{1-\cos\theta}{2}}{\frac{1+\cos\theta}{2}}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\text{x}\frac{1-\cos\theta}{1-\cos\theta}}$$

$$\tan\frac{1}{2}\theta = \frac{\sqrt{(1-\cos\theta)^2}}{\sqrt{1-\cos^2\theta}} = \frac{1-\cos\theta}{\sin\theta}$$

$$\boxed{\tan\frac{1}{2}\theta = \frac{1-\cos\theta}{\sin\theta}} \qquad \boxed{\tan\frac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta}}$$

$$\boxed{\cot\frac{1}{2}\theta = \frac{\sin\theta}{1-\cos\theta}} \qquad \boxed{\cot\frac{1}{2}\theta = \frac{1+\cos\theta}{\sin\theta}}$$

SUM and DIFFERENCE OF FUNCTIONS

$$\sin(x + y) = \sin x \cos y + \sin y \cos x$$

$$\sin(x - y) = \sin x \cos y - \sin y \cos x$$

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$$

$$x + y = \theta \qquad y = \frac{1}{2}(\theta - \phi)$$

$$x - y = \phi \qquad x = \frac{1}{2}(\theta + \phi)$$

$$\boxed{\sin\theta + \sin\phi = 2\sin\frac{1}{2}(\theta + \phi)\cos\frac{1}{2}(\theta - \phi)}$$

$$\sin(x + y) - \sin(x - y) = 2 \sin y \cos x$$

$$\boxed{\sin\theta - \sin\phi = 2\cos\frac{1}{2}(\theta + \phi)\sin\frac{1}{2}(\theta - \phi)}$$

$$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$$

$$\boxed{\cos\theta + \cos\phi = 2\cos\frac{1}{2}(\theta + \phi)\cos\frac{1}{2}(\theta - \phi)}$$

$$\cos(\theta - \phi) = \cos\theta \cos\phi + \sin\theta \sin\phi$$

$$\boxed{\cos\theta - \cos\phi = -2\sin\frac{1}{2}(\theta + \phi)\sin\frac{1}{2}(\theta - \phi)}$$

EXERCISES:

Represent as a product:
Page 132

1. $\sin 40 + \sin 20 = 2\sin\frac{1}{2}(40+20)\cos\frac{1}{2}(40-20)$

$$= 2\sin\frac{1}{2}(60)\cos\frac{1}{2}(20)$$

$$\boxed{= 2\sin 30\cos 10}$$

9. $\sin 32 + \cos 22$

$$\cos 22 = \sin(90-22) = \sin 68$$

$$\sin 32 + \sin 68 = 2\sin\frac{1}{2}(32+68)\cos\frac{1}{2}(32-68)$$

$$= 2\sin\frac{1}{2}(100)\cos\frac{1}{2}(-36)$$

$$= 2\sin 50\cos(-18)$$

$$\boxed{= 2\sin 50\cos 18}$$

Prove:
Page 133

1. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\sin(2\theta+\theta) = LS$$
$$\sin 2\theta\cos\theta + \cos 2\theta\sin\theta = LS$$
$$2\sin\theta\cos\theta(\cos\theta) + (\cos^2\theta - \sin^2\theta)\sin\theta = LS$$
$$2\sin\theta\cos^2\theta + (1-\sin^2\theta-\sin^2\theta)\sin\theta = LS$$
$$2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta = LS$$
$$2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta = LS$$
$$\boxed{3\sin\theta - 4\sin^3\theta = LS}$$

18. $\sin\theta + \cos\theta = \sqrt{2}\cos(\theta - 45)$

$$\sin(90 - \theta) = \cos\theta$$

$$\sin\theta + \sin(90 - \theta) = RS$$

$$2\sin\frac{1}{2}[\theta + (90 - \theta)]\cos\frac{1}{2}[\theta - (90 - \theta)] = RS$$

$$2\sin\frac{1}{2}(90)\cos\frac{1}{2}(\theta - 90 + \theta) = RS$$

$$2\sin 45\cos\frac{1}{2}(-90 + 2\theta) = RS$$

$$2\sin 45\cos(-45 + \theta) = RS$$

$$2\left(\frac{1}{\sqrt{2}}\right)\cos(\theta - 45) = RS$$

$$2\left(\frac{\sqrt{2}}{2}\right)\cos(\theta - 45) = RS$$

$$\boxed{\sqrt{2}\cos(\theta - 45) = RS}$$

20. $\dfrac{\sin\theta - \sin\phi}{\sin\theta + \sin\phi} = \dfrac{\tan\frac{1}{2}(\theta - \phi)}{\tan\frac{1}{2}(\theta + \phi)}$

$$\frac{2\sin\frac{1}{2}(\theta - \phi)\cos\frac{1}{2}(\theta + \phi)}{2\sin\frac{1}{2}(\theta + \phi)\cos\frac{1}{2}(\theta - \phi)} = RS$$

$$\tan\frac{1}{2}(\theta - \phi)\cot\frac{1}{2}(\theta + \phi) = RS$$

$$\boxed{\frac{\tan\frac{1}{2}(\theta - \phi)}{\tan\frac{1}{2}(\theta + \phi)} = RS}$$